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# THE LOGARITHMIC STRAIN SPACE DESCRIPTION

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**Abstract**—The ingredients of a plasticity description in the logarithmic strain space with respect to the reference configuration are discussed within the framework of the finite deformation theory, especially the features of logarithmic strain and the transformations between its time derivative, the rate of deformation tensor and the work-conjugate stresses. The logarithmic strain space description is then illustrated using three analytical examples of homogeneous finite deformation fields.

## 1. NOTATION

The summation convention is adopted on terms with repeated indices and the following definitions for second order tensors $\underset{\sim}{T}$ are introduced:

| | |
|---|---|
| $\underset{\sim}{T}^{T}$ | transpose of $\underset{\sim}{T}$ |
| $\underset{\sim}{T}^{-1}$ | inverse of $\underset{\sim}{T}$ |
| $\lvert T \rvert$ | determinant of $\underset{\sim}{T}$ |
| $T_{ij}$ | components of $\underset{\sim}{T}$ |
| $T_{kk} = T_{xx}+T_{yy}+T_{zz}$ | trace of $\underset{\sim}{T}$ |
| $\underset{\sim}{T}' = \underset{\sim}{T} - 1/3\, T_{kk}\, \underset{\sim}{1}$ | deviator of $\underset{\sim}{T}$ (the second order unit tensor is denoted by $\underset{\sim}{1}$) |
| $\lVert \underset{\sim}{T} \rVert = \sqrt{T_{ij} T_{ji}}$ | norm of $\underset{\sim}{T}$ |

Standard continuum mechanics notation is used, see Naghdi (1990):

| | |
|---|---|
| $\kappa_0$ or $\hat{\kappa}$ | reference or current configuration of body $B$ |
| $\underset{\sim}{X}$ or $\underset{\sim}{x}$ | position vector of a material point $P$ in $\kappa_0$ or $\hat{\kappa}$ |
| $t$ | time |
| $\underset{\sim}{x}(\underset{\sim}{X}, t)$ | motion of body $B$ in a reference description |
| $\underset{\sim}{F} = (\partial \underset{\sim}{x}/\partial \underset{\sim}{X})$ | deformation gradient, $\lvert \underset{\sim}{F} \rvert > 0$ |
| $\underset{\sim}{C} = \underset{\sim}{F}^{T}\underset{\sim}{F}$ or $\hat{\underset{\sim}{B}} = \underset{\sim}{F}\underset{\sim}{F}^{T}$ | s.p.d.† right or left Cauchy-Green tensor w.r.t.‡ $\kappa_0$ or $\hat{\kappa}$ |
| $\underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U} = \hat{\underset{\sim}{V}}\,\underset{\sim}{R}$ | polar decomposition of $\underset{\sim}{F}$ |
| $\underset{\sim}{R}$ | orthonormal rotation tensor |
| $\underset{\sim}{U} = \sqrt{\underset{\sim}{C}}$ or $\hat{\underset{\sim}{V}} = \sqrt{\hat{\underset{\sim}{B}}}$ | s.p.d. right or left stretch tensor w.r.t. $\kappa_0$ or $\hat{\kappa}$ |
| $\dot{\underset{\sim}{x}}$ | velocity of a material point $P$ |
| $\hat{\underset{\sim}{G}} = (\partial \dot{\underset{\sim}{x}}/\partial \underset{\sim}{x})$ | velocity gradient w.r.t. $\hat{\kappa}$ |
| $\hat{\underset{\sim}{D}} = \frac{1}{2}(\hat{\underset{\sim}{G}} + \hat{\underset{\sim}{G}}^{T})$ | rate of deformation tensor w.r.t. $\hat{\kappa}$, symmetric part of $\hat{\underset{\sim}{G}}$ |
| $\hat{\underset{\sim}{T}}$ | symmetric Cauchy stress w.r.t. $\hat{\kappa}$ |

Under superposed rigid body rotations the tensors with respect to the current configuration $\hat{\kappa}$, which are marked with a superscript *hat*, are altered, whereas the tensors with respect to the reference configuration $\kappa_0$ are not. The following tensors with respect to the reference configuration $\kappa_0$ can be defined by a back-rotation from the current configuration $\hat{\kappa}$:

| | |
|---|---|
| $\underset{\sim}{D} = \underset{\sim}{R}^{T}\hat{\underset{\sim}{D}}\underset{\sim}{R}$ | symmetric back-rotated rate of deformation tensor w.r.t. $\kappa_0$ |
| $\underset{\sim}{T} = \underset{\sim}{R}^{T}\hat{\underset{\sim}{T}}\underset{\sim}{R}$ | symmetric back-rotated stress w.r.t. $\kappa_0$ |

## 2. INTRODUCTION

The topic *logarithmic strain space description* combines the notions of *logarithmic strain* tensor and *strain space description* of plasticity. The logarithmic strain tensor was introduced by Hencky (1928); the features of its time derivative have been discussed by Hill (1970) and Hoger (1986). For the infinitesimal deformation theory the strain space description of plasticity was introduced by Il'iushin (1961). Within the thermodynamical framework of Green and Naghdi (1965) it has been generalized for the finite deformation theory by Naghdi and Trapp (1975).

† Symmetric positive definite.
‡ With respect to.

The distinction of the logarithmic strain $\underset{\sim}{\varepsilon}$ within the generalized finite strains with respect to the reference configuration $\kappa_0$, see Doyle and Ericksen (1956) and Hill (1968), is based upon the physical meaning of its trace $\varepsilon_{kk}$ and deviator $\underset{\sim}{\varepsilon}'$, which describe finite dilatation and finite distortion. Hence, by using logarithmic strains the finite dilatation and finite distortion can additively be decoupled, plastic incompressibility can be introduced by enforcing the plastic logarithmic strain $\underset{\sim}{\varepsilon}^p$ to be deviatoric and a von Mises type of yield function† can be expressed by the second invariant of $(\underset{\sim}{\varepsilon}'-\underset{\sim}{\varepsilon}^p)$, in analogy to the infinitesimal deformation theory.

The logarithmic strain space description is embedded in the framework of Naghdi and Trapp (1975), Casey and Naghdi (1983), Naghdi (1990), where we replace the total and plastic Green–Lagrange strain tensors $\underset{\sim}{E}$ and $\underset{\sim}{E}^p$ by the logarithmic strain tensors $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{\varepsilon}^p$, respectively. Moreover, following Hill (1968) and Hoger (1987), a stress which is work-conjugate to the logarithmic strain rate is introduced and its relation to the Cauchy stress is specified.

Three analytical examples are presented in order to illustrate the logarithmic strain space description of plasticity. They are based on homogeneous finite deformation kinematics. In the first and second examples of finite tension and finite shear, the additive decoupling of finite dilatation and finite distortion is demonstrated. Furthermore, the second example clarifies the physical basis of finite isotropic shear. A finite shear deformation applied on isotropic elasto–plastic material should only result in associated shear stress components. It should not result in tensile stress components as is the case in the rectilinear shear examples of Lehmann (1972), Dienes (1979), Eterovic and Bathe (1990), Weber and Anand (1990) and many others. The third example covers a homogeneous plane deformation field in conjunction with rigid ideal plasticity, such that the principal directions of the logarithmic strain $\underset{\sim}{\varepsilon}$ with respect to the reference configuration $\kappa_0$ change. Hence, the logarithmic strain rate and the rate of deformation are no longer identical and so also their work-conjugate stresses. This example may serve as a link between the logarithmic strain space description and solutions which are based upon the method of characteristics, see Prandtl (1920) and Geiringer (1930).

## 3. LOGARITHMIC STRAIN AND ITS FEATURES

The symmetric logarithmic strain tensors of Hencky (1928) with respect to the reference $\kappa_0$ and current configurations $\hat{\kappa}$ are defined by

$$\underset{\sim}{\varepsilon} = \ln(\underset{\sim}{U}) = \tfrac{1}{2}\ln(\underset{\sim}{C}) \quad \text{and} \quad \hat{\underset{\sim}{\varepsilon}} = \ln(\hat{\underset{\sim}{V}}) = \tfrac{1}{2}\ln(\hat{\underset{\sim}{B}}),$$

respectively. They obey

$$\underset{\sim}{\varepsilon} = \underset{\sim}{R}^{\mathrm{T}}\,\hat{\underset{\sim}{\varepsilon}}\,\underset{\sim}{R}$$

and their numerical calculation involves:

- the transformation to principal axes of $\underset{\sim}{U}$ ($\underset{\sim}{C}$, $\hat{\underset{\sim}{V}}$ or $\hat{\underset{\sim}{B}}$)

$$\bar{U}_{IJ} = Q_{KI}\,U_{KL}\,Q_{LJ}: \quad \begin{bmatrix} U_1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & U_3 \end{bmatrix} = [\mathrm{Q}^{\mathrm{T}}] \begin{bmatrix} \mathrm{U}_{xx} & \mathrm{U}_{xy} & \mathrm{U}_{zx} \\ \mathrm{U}_{xy} & \mathrm{U}_{yy} & \mathrm{U}_{yz} \\ \mathrm{U}_{zx} & \mathrm{U}_{yz} & \mathrm{U}_{zz} \end{bmatrix} [\mathrm{Q}], \tag{1}$$

- the application of the logarithm to the principal values of $\underset{\sim}{U}$ ($\underset{\sim}{C}$, $\hat{\underset{\sim}{V}}$ or $\hat{\underset{\sim}{B}}$)

† In a von Mises type of yield function plastic flow occurs at a certain elastic distortional energy, see von Mises (1928).

$$\bar{\varepsilon}_{IJ} = \ln(\bar{U}_{IJ}): \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} = \begin{bmatrix} \ln(U_1) & 0 & 0 \\ 0 & \ln(U_2) & 0 \\ 0 & 0 & \ln(U_3) \end{bmatrix} \tag{2}$$

- and the reverse transformation of the principal values of $\underset{\sim}{\varepsilon}$ (or $\hat{\underset{\sim}{\varepsilon}}$)

$$\varepsilon_{IJ} = \mathrm{Q}_{IK}\,\bar{\varepsilon}_{KL}\,\mathrm{Q}_{JL}: \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{zx} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} = [\mathrm{Q}] \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} [\mathrm{Q}^{\mathrm{T}}], \tag{3}$$

where the *overbar* denotes the tensor components in the principal co-ordinate system and where the $\mathrm{Q}_{IJ}$ or [Q] denote the components of the orthonormal rotation to principal axes.

### 3.1. *Dilatation and distortion*

For the finite deformation theory the determinant $|\underset{\sim}{\mathrm{F}}|$ of the deformation gradient is a measure of dilatation. Since

$$\varepsilon_{KK} = \hat{\varepsilon}_{kk} = \ln(|\underset{\sim}{\mathrm{F}}|)$$

holds, the traces of the logarithmic strains are also measures of finite dilatation, see Hencky (1928).

For the finite deformation theory

$$\underset{\sim}{\mathrm{U}}^* = \frac{\underset{\sim}{\mathrm{U}}}{\sqrt[3]{|\underset{\sim}{\mathrm{U}}|}} \quad \text{and} \quad \hat{\underset{\sim}{\mathrm{V}}}^* = \frac{\hat{\underset{\sim}{\mathrm{V}}}}{\sqrt[3]{|\hat{\underset{\sim}{\mathrm{V}}}|}}$$

are measures of distortion†, see Flory (1961). Since

$$\underset{\sim}{\varepsilon}' = \ln(\underset{\sim}{\mathrm{U}}^*) \quad \text{and} \quad \hat{\underset{\sim}{\varepsilon}}' = \ln(\hat{\underset{\sim}{\mathrm{V}}}^*)$$

hold, the deviators of the logarithmic strains are also measures of finite distortion.

### 3.2. *Logarithmic strain rates and back-rotated rate of deformation*

In order to elucidate the material time derivative we specify the relations between the base vectors in the current $\underline{\hat{\mathrm{e}}}_i$ and reference configuration $\underline{\mathrm{E}}_i$ by

$$\underline{\hat{\mathrm{e}}}_i = \underset{\sim}{\mathrm{R}}\underline{\mathrm{E}}_i, \tag{4}$$

so that they are co-rotating with respect to the material. From time differentiation and the inverse of (4) we have

$$\dot{\underline{\hat{\mathrm{e}}}}_i = \underset{\sim}{\omega}\underline{\hat{\mathrm{e}}}_i \quad \text{with} \quad \underset{\sim}{\omega} = \dot{\underset{\sim}{\mathrm{R}}}\underset{\sim}{\mathrm{R}}^{\mathrm{T}}. \tag{5}$$

Hence, the materially co-rotated time derivatives of second order tensors with respect to $\kappa_0$ and $\hat{\kappa}$,

$$\underset{\sim}{\mathrm{T}} = \mathrm{T}_{IJ}\underline{\mathrm{E}}_I \otimes \underline{\mathrm{E}}_J \quad \text{and} \quad \hat{\underset{\sim}{\mathrm{T}}} = \hat{\mathrm{T}}_{ij}\underline{\hat{\mathrm{e}}}_i \otimes \underline{\hat{\mathrm{e}}}_j,$$

are labelled by a superscript *triangle* and are given by

† By definition $|\underset{\sim}{\mathrm{U}}^*| = |\hat{\underset{\sim}{\mathrm{V}}}^*| = 1$.

$$\overset{\Delta}{\underset{\sim}{T}} = \dot{T}_{IJ}\underline{E}_I \otimes \underline{E}_J \quad \text{and} \quad \overset{\Delta}{\underset{\sim}{T}} = \dot{\hat{T}}_{ij}\hat{\underline{e}}_i \otimes \hat{\underline{e}}_j + \hat{T}_{ij}\dot{\hat{\underline{e}}}_i \otimes \hat{\underline{e}}_j + \hat{T}_{ij}\hat{\underline{e}}_i \otimes \dot{\hat{\underline{e}}}_j,$$

where $\otimes$ denotes the dyadic product. These are given in tensor notation, respectively, by

$$\overset{\Delta}{\underset{\sim}{T}} = \dot{\underset{\sim}{T}} \quad \text{and} \quad \overset{\Delta}{\hat{\underset{\sim}{T}}} = \dot{\hat{\underset{\sim}{T}}} + \hat{\underset{\sim}{T}}\underset{\sim}{\omega} - \underset{\sim}{\omega}\hat{\underset{\sim}{T}} = \underset{\sim}{R}(\underset{\sim}{R}^{T}\hat{\underset{\sim}{T}}\underset{\sim}{R})^{\cdot}\underset{\sim}{R}^{T}.$$

Therefore, the materially co-rotated logarithmic strain rate with respect to $\hat{\kappa}$ and the logarithmic strain rate with respect to $\kappa_0$ obey

$$\overset{\Delta}{\hat{\underset{\sim}{\varepsilon}}} = \underset{\sim}{R}\dot{\underset{\sim}{\varepsilon}}\underset{\sim}{R}^{T}. \tag{6}$$

The back-rotated rate of deformation tensor with respect to $\kappa_0$ can be expressed by

$$\underset{\sim}{D} = \tfrac{1}{2}(\dot{\underset{\sim}{U}}\underset{\sim}{U}^{-1} + \underset{\sim}{U}^{-1}\dot{\underset{\sim}{U}}).$$

In order to derive the relations between $\dot{\underset{\sim}{\varepsilon}}$ and $\underset{\sim}{D}$ we specify their components in the principal co-ordinate system† of eqn (1)

$$\dot{\bar{\varepsilon}}_{IJ} = \begin{bmatrix} \dfrac{\dot{U}_1}{U_1} & \Omega_3 \ln\left(\dfrac{U_1}{U_2}\right) & \Omega_2 \ln\left(\dfrac{U_3}{U_1}\right) \\ \Omega_3 \ln\left(\dfrac{U_1}{U_2}\right) & \dfrac{\dot{U}_2}{U_2} & \Omega_1 \ln\left(\dfrac{U_2}{U_3}\right) \\ \Omega_2 \ln\left(\dfrac{U_3}{U_1}\right) & \Omega_1 \ln\left(\dfrac{U_2}{U_3}\right) & \dfrac{\dot{U}_3}{U_3} \end{bmatrix},$$

$$\bar{D}_{IJ} = \begin{bmatrix} \dfrac{\dot{U}_1}{U_1} & \dfrac{\Omega_3}{2}\left(\dfrac{U_1}{U_2} - \dfrac{U_2}{U_1}\right) & \dfrac{\Omega_2}{2}\left(\dfrac{U_3}{U_1} - \dfrac{U_1}{U_3}\right) \\ \dfrac{\Omega_3}{2}\left(\dfrac{U_1}{U_2} - \dfrac{U_2}{U_1}\right) & \dfrac{\dot{U}_2}{U_2} & \dfrac{\Omega_1}{2}\left(\dfrac{U_2}{U_3} - \dfrac{U_3}{U_2}\right) \\ \dfrac{\Omega_2}{2}\left(\dfrac{U_3}{U_1} - \dfrac{U_1}{U_3}\right) & \dfrac{\Omega_1}{2}\left(\dfrac{U_2}{U_3} - \dfrac{U_3}{U_2}\right) & \dfrac{\dot{U}_3}{U_3} \end{bmatrix} \tag{7}$$

and denote the components of the rotation velocity of the principal axes by $\Omega_1$, $\Omega_2$, $\Omega_3$ analogous to $\underset{\sim}{\omega}$ in (5).

### 3.3. *Fourth order kinematical transformation tensors*

The transformations between the back-rotated rate of deformation tensor $\underset{\sim}{D}$ and the logarithmic strain rate $\dot{\underset{\sim}{\varepsilon}}$ with respect to the reference configuration $\kappa_0$

$$\underset{\sim}{D} = \underset{\vee}{\alpha}\dot{\underset{\sim}{\varepsilon}}, \quad D_{IJ} = \alpha_{IJKL}\dot{\varepsilon}_{KL} \quad \text{and} \quad \dot{\underset{\sim}{\varepsilon}} = \underset{\vee}{\beta}\underset{\sim}{D}, \quad \dot{\varepsilon}_{IJ} = \beta_{IJKL}D_{KL} \tag{8}$$

are specified by the fourth order kinematical transformation tensors $\underset{\vee}{\alpha}$ and $\underset{\vee}{\beta}$, which are inverse to each other

$$\underset{\vee}{\alpha}\underset{\vee}{\beta} = \underset{\vee}{\beta}\underset{\vee}{\alpha} = \underset{\vee}{1},$$

where $\underset{\vee}{1}$ denotes the fourth order unit tensor. The $\underset{\vee}{\alpha}$ and $\underset{\vee}{\beta}$ obey the symmetries

† Components specified in a principal co-ordinate system are marked by an *overbar*, cf. eqns (1)–(3).

$$\beta_{IJKL} = \beta_{JIKL} = \beta_{IJLK} = \beta_{KLIJ}. \tag{9}$$

In the principal co-ordinate system† of (1) the non-zero components of $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ follow from (7) and are given by

$$\bar{\alpha}_{1111} = \bar{\alpha}_{2222} = \bar{\alpha}_{3333} = \bar{\beta}_{1111} = \bar{\beta}_{2222} = \bar{\beta}_{3333} = 1$$

$$2\bar{\alpha}_{1212} = 2\bar{\alpha}_{1221} = 2\bar{\alpha}_{2112} = 2\bar{\alpha}_{2121} = \frac{1}{2\bar{\beta}_{1212}} = \cdots = \frac{U_1/U_2 - U_2/U_1}{2\ln(U_1/U_2)}$$

$$2\bar{\alpha}_{2323} = \cdots = \frac{1}{2\bar{\beta}_{2323}} = \cdots = \frac{U_2/U_3 - U_3/U_2}{2\ln(U_2/U_3)}$$

$$2\bar{\alpha}_{3131} = \cdots = \frac{1}{2\bar{\beta}_{3131}} = \cdots = \frac{U_3/U_1 - U_1/U_3}{2\ln(U_3/U_1)}. \tag{10}$$

The factors of the shear components (10) are of the type

$$\frac{q - 1/q}{2\ln(q)} \tag{11}$$

when denoting $U_1/U_2$ or $U_2/U_3$ or $U_3/U_1$ by $q$. If two principal values of $\underset{\sim}{U}$ are of the same magnitude, i.e. $q = 1$, then eqn (11) becomes indeterminate, but its limit is well defined

$$\lim_{q \to 1} \frac{q - 1/q}{2\ln(q)} = 1.$$

The kinematical transformation tensor components with respect to arbitrary Cartesian co-ordinates

$$\alpha_{IJKL} = Q_{IM}\, Q_{JN}\, Q_{KO}\, Q_{LP}\, \bar{\alpha}_{MNOP} \quad \text{and} \quad \beta_{IJKL} = Q_{IM}\, Q_{JN}\, Q_{KO}\, Q_{LP}\, \bar{\beta}_{MNOP}$$

follow from the reverse rotation of the principal components [eqns (10)], where the $Q_{IM}$ are defined by eqn (3). The transformations between the rate of deformation tensor and the materially co-rotated logarithmic strain rate with respect to $\hat{\kappa}$

$$\hat{\underset{\sim}{D}} = \hat{\underset{\sim}{\alpha}}\overset{\Delta}{\hat{\underset{\sim}{\varepsilon}}} \quad \text{and} \quad \overset{\Delta}{\hat{\underset{\sim}{\varepsilon}}} = \hat{\underset{\sim}{\beta}}\hat{\underset{\sim}{D}} \tag{12}$$

are given by the fourth order kinematical transformation tensors

$$\hat{\alpha}_{ijkl} = R_{iI} R_{jJ} R_{kK} R_{lL}\, \alpha_{IJKL} \quad \text{and} \quad \hat{\beta}_{ijkl} = R_{iI} R_{jJ} R_{kK} R_{lL} \beta_{IJLK},$$

which are inverse to each other and obey the symmetries [eqn (9)]. The $R_{iI}$ denote the components of the rotation tensor $\underset{\sim}{R}$. The fourth order transformation tensors become fourth order unit tensors

$$\hat{\underset{\sim}{\alpha}} = \hat{\underset{\sim}{\beta}} = \underset{\sim}{\alpha} = \underset{\sim}{\beta} = \underset{\sim}{1}, \tag{13}$$

if the principal values of $\underset{\sim}{U}$ are identical, $U_1 = U_2 = U_3$, if the generalized finite strains with

† Components specified in a principal co-ordinate system are marked by an *overbar*, cf. eqns (1)–(3).

respect to the reference configuration $\kappa_0$ have constant principal directions† or for the infinitesimal deformation theory‡.

According to Hill (1968), MacVean (1968) and Hoger (1987) we define§ the stresses which are work-conjugate to the logarithmic strain rates with respect to $\kappa_0$ and $\hat{\kappa}$ by

$$\underset{\sim}{\sigma} = \underset{\vee}{\alpha}\underset{\sim}{\mathrm{T}} \quad \text{and} \quad \hat{\underset{\sim}{\sigma}} = \hat{\underset{\vee}{\alpha}}\hat{\underset{\sim}{\mathrm{T}}}, \tag{14}$$

respectively. Furthermore,

$$\underset{\sim}{\mathrm{T}} = \underset{\vee}{\beta}\underset{\sim}{\sigma}, \quad \hat{\underset{\sim}{\mathrm{T}}} = \hat{\underset{\vee}{\beta}}\hat{\underset{\sim}{\sigma}} \quad \text{and} \quad \hat{\underset{\sim}{\sigma}} = \underset{\sim}{\mathrm{R}}\underset{\sim}{\sigma}\underset{\sim}{\mathrm{R}}^{\mathrm{T}} \tag{15}$$

hold. The stresses $\underset{\sim}{\sigma}$, $\underset{\sim}{\mathrm{T}}$, $\hat{\underset{\sim}{\sigma}}$ and $\hat{\underset{\sim}{\mathrm{T}}}$ are work-conjugate to the deformation rate tensors $\dot{\underset{\sim}{\varepsilon}}$, $\underset{\sim}{\mathrm{D}}$, $\overset{\Delta}{\underset{\sim}{\varepsilon}}$ and $\hat{\underset{\sim}{\mathrm{D}}}$, respectively:

$$\sigma_{IJ}\dot{\varepsilon}_{JI} = \mathrm{T}_{KL}\mathrm{D}_{LK} = \hat{\sigma}_{ij}\overset{\Delta}{\varepsilon}_{ji} = \hat{\mathrm{T}}_{kl}\hat{\mathrm{D}}_{lk}.$$

In principle, the tensor transformations [eqns (8), (12), (14) and (15)] only apply for the deviators, as the first invariants, i.e. the traces,

$$\dot{\varepsilon}_{KK} = \mathrm{D}_{KK} = \overset{\Delta}{\varepsilon}_{kk} = \hat{\mathrm{D}}_{kk} \quad \text{or} \quad \sigma_{KK} = \mathrm{T}_{KK} = \hat{\sigma}_{kk} = \hat{\mathrm{T}}_{kk},$$

are equal and express the rate of dilatation or three times the negative hydrostatic pressure, respectively.

## 4. CONSTITUTIVE EQUATIONS

We focus on the purely mechanical aspects of the general thermomechanical framework presented in Green and Naghdi (1965). We consider plasticity descriptions in strain spaces with respect to the reference configuration $\kappa_0$ and replace the total and plastic Green–Lagrange strain tensors $\underset{\sim}{\mathrm{E}}$ and $\underset{\sim}{\mathrm{E}}^p$ in Naghdi and Trapp (1975), Casey and Naghdi (1983) and Naghdi (1990) by the logarithmic strain tensors $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{\varepsilon}^p$, respectively. The logarithmic strain space is a general description in the sense that elastic and plastic anisotropy, non-associated flow rules and complex scalar or tensorial hardening rules can be described. However, in view of the examples presented below we specify the general constitutive description for the case of isotropic elasto-plasticity with a von Mises type of yield function, an associated flow rule and isotropic hardening. Moreover, we specify the hardening function $\bar{\sigma}(\bar{\varepsilon}^p)$, depicted in Fig. 1, as a linear function of the equivalent plastic strain $\bar{\varepsilon}^p$.

In the logarithmic strain space description the independent variables are the total logarithmic strain $\underset{\sim}{\varepsilon}$, the plastic logarithmic strain¶ $\underset{\sim}{\varepsilon}^p$ and a hardening parameter. In the

† For constant principal directions the rotation velocity of the principal axes vanishes, $\Omega_1 = \Omega_2 = \Omega_3 = 0$. Hence, from (7), the logarithmic strain rate and the rate of deformation tensor are identical, i.e. eqn (13) holds.

‡ Infinitesimal deformations can mathematically be expressed by the expansion $\underset{\sim}{\mathrm{U}} = \underset{\sim}{1} + \underset{\sim}{u} + O(\varepsilon^2)$ of the right stretch tensor, where the components $u_{IJ} = O(\varepsilon)$ of the first order term $\underset{\sim}{u}$ are of the order of small numbers, $\varepsilon \ll 1$, compared to unity. The corresponding expansions of the principal values $U_1 = 1 + u_1$, $U_2 = 1 + u_2$ and $U_3 = 1 + u_3$ introduced into (10b), (10c) and (10d) yield $\bar{\alpha}_{IJKL} = \bar{\beta}_{IJKL} = 1/2(\delta_{IK}\delta_{JL} + \delta_{IL}\delta_{JK}) + O(\varepsilon^2)$. Hence, for the infinitesimal deformation theory the logarithmic strain rate and the rate of deformation tensor are identical, i.e. eqn (13) holds.

§ The symmetry $\alpha_{IJKL} = \alpha_{KLIJ}$ is applied for the stress definition.

¶ The evolution of $\underset{\sim}{\varepsilon}^p$ and $\bar{\varepsilon}^p$ is traced by the time integral of the rates $\dot{\underset{\sim}{\varepsilon}}^p$ and $\dot{\bar{\varepsilon}}^p$, respectively.

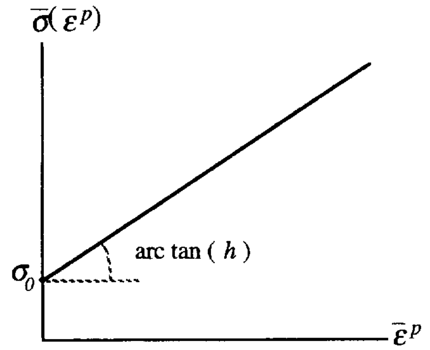


Fig. 1. Linear hardening function $\bar{\sigma}(\bar{\varepsilon}^p)$.

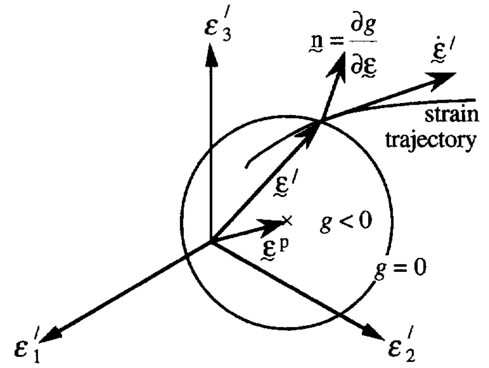


Fig. 2. Von Mises type of yield function in the deviatoric plane of the space of principal logarithmic strains $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$.

examples presented below the hardening parameter is specified by the equivalent plastic strain¶ $\bar{\varepsilon}^p$. A yield function of von Mises type is given by

$$g(\underset{\sim}{\varepsilon}, \underset{\sim}{\varepsilon}^p, \bar{\varepsilon}^p) = \| \underset{\sim}{\varepsilon}' - \underset{\sim}{\varepsilon}^p \| - \sqrt{\frac{2}{3}} \frac{\bar{\sigma}(\bar{\varepsilon}^p)}{2G}, \tag{16}$$

where $\| \cdot \|$, $\underset{\sim}{\varepsilon}'$, $\bar{\sigma}(\bar{\varepsilon}^p)$ and $G$ denote the norm, the deviator of a second order tensor (both defined in Section 2), the hardening function (see Fig. 1) and the shear modulus, respectively. The gradient to the yield function [eqn (16)] and the projection of the total logarithmic strain rate $\dot{\underset{\sim}{\varepsilon}}$ on the gradient are given by

$$\frac{\partial_g}{\partial \underset{\sim}{\varepsilon}} = \frac{\underset{\sim}{\varepsilon}' - \underset{\sim}{\varepsilon}^p}{\|\underset{\sim}{\varepsilon}' - \underset{\sim}{\varepsilon}^p\|} = \underset{\sim}{n} \quad \text{and} \quad \hat{g} = \frac{\partial_g}{\partial \varepsilon_{KL}} \dot{\varepsilon}_{KL} = n_{LK} \dot{\varepsilon}_{KL}, \tag{17}$$

respectively. The gradient (17) (first part), is denoted by $\underset{\sim}{n}$, because it has unit norm, $\|\underset{\sim}{n}\| = 1$; furthermore, it is deviatoric, $n_{KK} = 0$. The plastic flow rule or, equivalently, the plastic logarithmic strain rate is

$$\dot{\underset{\sim}{\varepsilon}}^p = \begin{cases} 0 & \text{if } g < 0 & & \text{(elastic state)} \\ 0 & \text{if } g = 0 & \text{and } \hat{g} < 0 & \text{(unloading)} \\ 0 & \text{if } g = 0 & \text{and } \hat{g} = 0 & \text{(neutral loading)} \\ \frac{3G}{3G+h} \hat{g} \underset{\sim}{n} & \text{if } g = 0 & \text{and } \hat{g} > 0 & \text{(loading)} \end{cases} \tag{18}$$

and, therefore, plastic flow can only occur if the total logarithmic strain is located at the yield surface, $g = 0$, and is pushing outwards, $\hat{g} > 0$, as depicted in Fig. 2. The hardening slope $h$, used in (18), is defined by Fig. 1, and the hardening law is given by

$$\dot{\varepsilon}^p = \sqrt{\tfrac{2}{3}}\,\|\dot{\boldsymbol{\varepsilon}}^p\| = \sqrt{\tfrac{2}{3}}\,n_{IJ}\dot{\varepsilon}^p_{JI}. \tag{19}$$

With eqns (16)–(19) the consistency condition, $\dot{g} = 0$, is fulfilled.

It should be noted that no stress is used to describe the plasticity. However, the work-conjugate stress with respect to the reference configuration $\kappa_0$, written for the trace and the deviator, follows† from

$$\sigma_{KK} = \frac{E}{1-2\nu}\varepsilon_{KK} \quad \text{and} \quad \boldsymbol{\sigma}' = \frac{E}{1+\nu}(\boldsymbol{\varepsilon}' - \boldsymbol{\varepsilon}^p), \tag{20}$$

where Poisson's ratio is denoted by $\nu$ and Young's modulus by $E = 2G(1+\nu)$.

## 5. FINITE TENSION WITH ISOTROPIC ELASTO-PLASTICITY

Consider the homogeneous finite deformation field‡

$$x = \lambda\mu^2 X, \quad y = \frac{\lambda}{\mu}Y, \quad z = \frac{\lambda}{\mu}Z, \tag{21}$$

where $x, y, z$ denote the co-ordinates in the current configuration $\hat{\kappa}$ and $X, Y, Z$ the co-ordinates in the reference configuration $\kappa_0$. The kinematic parameters $\lambda$ and $\mu$ describe the dilatation and the distortion, respectively. For the deformation field [eqns (21)] the components of the deformation gradient $\mathbf{F}$ and the right or left stretch tensors $\mathbf{U}$ or $\mathbf{V}$ are identical, since $\mathbf{R} = \mathbf{1}$ holds

$$F_{iJ} = U_{IJ} = V_{ij} = \begin{bmatrix} \lambda\mu^2 & 0 & 0 \\ 0 & \lambda/\mu & 0 \\ 0 & 0 & \lambda/\mu \end{bmatrix}. \tag{22}$$

The logarithmic strain tensors are also identical

$$\varepsilon_{IJ} = \hat{\varepsilon}_{ij} = \ln(\lambda)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \ln(\mu)\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{23}$$

The determinant of the deformation gradient [eqn (22)] and the traces of the logarithmic strains [eqn (23)] are

$$|\mathbf{F}| = \lambda^3 \quad \text{and} \quad \varepsilon_{KK} = \hat{\varepsilon}_{kk} = 3\ln(\lambda).$$

For the deformation field [eqn (21)] the generalized finite strains with respect to the reference configuration $\kappa_0$ have constant principal directions, so that eqn (13) holds. In the following we consider a monotonic loading path and a material behaviour as depicted in Fig. 1.

† In general, the stress is given by the derivative of a strain energy function with respect to the total strain.

‡ For simplicity the deformation field [eqns (21)] is specified without any rigid body rotation, i.e. $\mathbf{R} = \mathbf{1}$. But an arbitrary rotation, $\mathbf{R} \neq \mathbf{1}$, can be used. As long as the generalized finite strains with respect to the reference configuration $\kappa_0$ remain unchanged the plasticity description also remains unchanged.

5.1. *Elastic range*

The kinematic parameters

$$\lambda = \exp\left(\frac{1-2\nu}{3}\tau\right), \quad \mu = \exp\left(\frac{1+\nu}{3}\tau\right) \tag{24}$$

are specified as exponential functions of the monotonically increasing ($\tau \geqslant 0$, tension) or decreasing ($\tau \leqslant 0$, compression) load parameter $\tau$ and of Poisson's ratio $\nu$. For elasticity $\tau$ is restricted to

$$0 \leqslant |\tau| \leqslant \frac{\sigma_0}{E},$$

where $\sigma_0$ denotes the yield stress and $E$ Young's modulus. For metals we have $|\tau| \ll 1$, as $\sigma_0$ is of several orders of magnitude smaller than $E$. With eqns (24) the components of the logarithmic total and plastic strain tensors then are

$$\varepsilon_{IJ} = \hat{\varepsilon}_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\nu \end{bmatrix} \tau, \quad \varepsilon^{p}_{IJ} = \hat{\varepsilon}^{p}_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

As a consequence of eqn (13) the components of the stresses $\hat{\underset{\sim}{T}}$, $\underset{\sim}{T}$, $\hat{\underset{\sim}{\sigma}}$ and $\underset{\sim}{\sigma}$ are identical and follow from eqns (20), cf. Fig. 1,

$$\hat{T}_{ij} = T_{IJ} = \sigma_{IJ} = \hat{\sigma}_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} E\tau.$$

5.2. *Plastic range*

For plasticity $\tau$ is restricted to

$$\frac{\sigma_0}{E} < |\tau|$$

and the kinematic parameters $\lambda$ and $\mu$ are specified by

$$\lambda = \exp\left(\frac{(1-2\nu)(h\tau \pm \sigma_0)}{3(E+h)}\right), \quad \mu = \exp\left(\frac{\{3E+2(1+\nu)h\}\tau \mp (1-2\nu)\sigma_0}{3(E+h)}\right), \tag{25}$$

where the upper sign corresponds to tension ($\tau > 0$) and the lower sign to compression ($\tau < 0$). With eqns (25) the components of the logarithmic total and plastic strain tensors follow from eqn (23) and the time integral of eqn (18)

$$\varepsilon_{IJ} - \varepsilon^{p}_{IJ} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\nu \end{bmatrix} \frac{h\tau \pm \sigma_0}{E+h}, \quad \varepsilon^{p}_{IJ} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \frac{E\tau \mp \sigma_0}{2(E+h)}.$$

Again, as a consequence of eqn (13) the components of the stresses $\hat{\underset{\sim}{T}}$, $\underset{\sim}{T}$, $\hat{\underset{\sim}{\sigma}}$, $\underset{\sim}{\sigma}$ are identical and follow from eqns (20), cf. Fig. 1,

$$\sigma_{IJ} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{E(h\tau \pm \sigma_0)}{E+h}.$$

The experimental determination of hardening behaviour is often based upon uniaxial tension tests. The above example may therefore be generalized to hardening-dependent hardening slopes $h(\bar{\varepsilon}^p)$. In the expressions for the current subsection the constant hardening slopes $h$ need then to be replaced by $h(\bar{\varepsilon}^p)$ and functions of $h(\bar{\varepsilon}^p)$ by the corresponding integrals.

## 6. FINITE SHEAR WITH ISOTROPIC ELASTO-PLASTICITY

Consider the plane finite shear deformation field

$$x = \frac{(1+\mu^2)\cos\varphi + (1-\mu^2)\sin\varphi}{2\mu} X - \frac{(1-\mu^2)\cos\varphi + (1+\mu^2)\sin\varphi}{2\mu} Y,$$

$$y = -\frac{(1-\mu^2)\cos\varphi - (1+\mu^2)\sin\varphi}{2\mu} X + \frac{(1+\mu^2)\cos\varphi - (1-\mu^2)\sin\varphi}{2\mu} Y, \qquad (26)$$

where the components of the rotation tensor $\underset{\sim}{R}$, the right stretch tensor $\underset{\sim}{U}$ and the logarithmic strain tensor with respect to the reference configuration $\underset{\sim}{\varepsilon}$ are

$$R_{iI} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}, \quad U_{IJ} = \frac{1}{2\mu}\begin{bmatrix} \mu^2+1 & \mu^2-1 \\ \mu^2-1 & \mu^2+1 \end{bmatrix}, \quad \varepsilon_{IJ} = \ln(\mu)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

respectively. For an angle of rigid body rotation

$$\varphi = \arctan\left(\frac{1-\mu^2}{1+\mu^2}\right)$$

or, equivalently,

$$\sin\varphi = \frac{1-\mu^2}{\sqrt{2(1+\mu^4)}}, \quad \cos\varphi = \frac{1+\mu^2}{\sqrt{2(1+\mu^4)}},$$

the shear deformation field (26) is given by

$$x = \frac{(1+\mu^4)X - (1-\mu^4)Y}{\sqrt{2}\mu\sqrt{(1+\mu^4)}}, \quad y = \frac{\sqrt{2}\mu Y}{\sqrt{(1+\mu^4)}}. \qquad (27)$$

A square with side lengths $a$, which undergoes the deformation [eqns (27)] deforms to a parallelogram with constant side lengths

$$b = \sqrt{\frac{\mu^2 + \mu^{-2}}{2}}\, a$$

as depicted in Fig. 3. For torsion tests on thin-walled isotropic tubes this behaviour should lead to decreasing length and increasing diameter of the tube, as experimentally observed by Müller and Pöhlandt (1992) for a magnesium specimen. Using (24) and the monotonically increasing (tension) or decreasing (compression) load parameter $\tau$ the non-zero components of $\underset{\sim}{\varepsilon}$ can be expressed by

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1+\nu}{3}\tau,$$

of $\varepsilon^p$ by

$$\varepsilon^p_{xy} = \varepsilon^p_{yx} = \begin{cases} 0 & \text{for} \quad 0 \leqslant |\tau| \leqslant \sqrt{3}\sigma_0/E \quad \text{(elastic range)} \\ \dfrac{1+\nu}{3(E+h)}(E\tau \mp \sqrt{3}\sigma_0) & \text{for} \quad \sqrt{3}\sigma_0/E < |\tau| \quad \text{(plastic range)} \end{cases}$$

and of $\underset{\sim}{\sigma}$ or $\underset{\sim}{T}$, cf. Fig. 1, by

$$\sigma_{xy} = T_{yx} = \begin{cases} \frac{1}{3}E\tau & \text{for} \quad 0 \leqslant |\tau| \leqslant \sqrt{3}\sigma_0/E \quad \text{(elastic range)} \\ \dfrac{E}{3(E+h)}(h\tau \pm \sqrt{3}\sigma_0) & \text{for} \quad \sqrt{3}\sigma_0/E < |\tau| \quad \text{(plastic range),} \end{cases}$$

where the upper sign corresponds to tension ($\tau > 0$) and the lower sign to compression ($\tau < 0$). Since eqn (13) holds, the stresses $\underset{\sim}{\sigma}$ and $\underset{\sim}{T}$ are identical.

On the other hand and in conjunction with the large body of finite element literature, Lehmann (1972), Dienes (1979), Eterovic and Bathe (1990), Weber and Anand (1990) and many others, misuse the rectilinear shear, depicted in Fig. 4, in order to find the most suitable time derivative of a tensor with respect to the current configuration $\hat{\kappa}$. Rectilinear shear, in which an initially square shape is deformed into a parallelogram with side lengths having pairs of $a$ and $c$, is a misleading generalization of the small deformation example.

A stress field of pure shear, namely, has two planes of symmetry and, therefore, the corresponding deformation field should also have two planes of symmetry. This is the case for the deformation field [eqns (27)] depicted in Fig. 3, but not for the rectilinear shear example of Fig. 4. For $\varphi = 0$, i.e. no rigid body rotation, the deformation field [eqns (26)] is depicted by bold lines under angles of 45° in Fig. 5. A co-ordinate rotation of 45° then reveals the tension/compression kinematics, depicted in Fig. 5 with dashed lines, in conjunction with the corresponding rotation of the stress components depicted in Fig. 6.

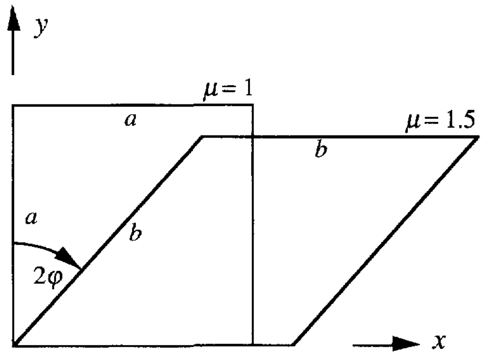


Fig. 3. Homogeneous equi-volumetrical finite shear deformation resulting in parallelograms with equal side lengths $b$.

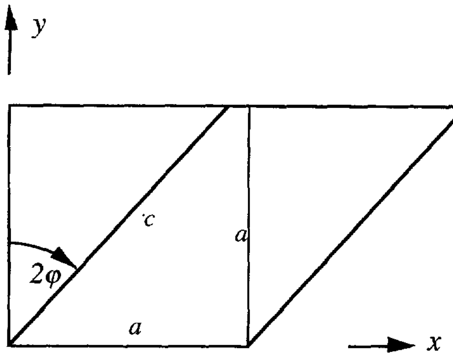


Fig. 4. Rectilinear shear, a misleading generalization of a small deformation example, resulting in parallelograms with side lengths having pairs of $a$ and $c$.

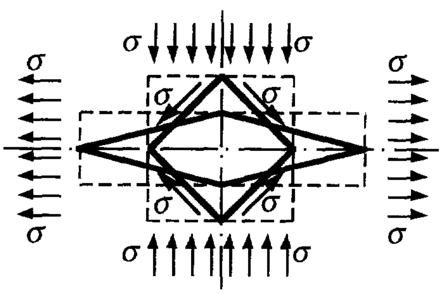

Fig. 5. Kinematics of finite shear with horizontal and vertical planes of symmetry viewed as tension/compression or under 45° as pure shear.

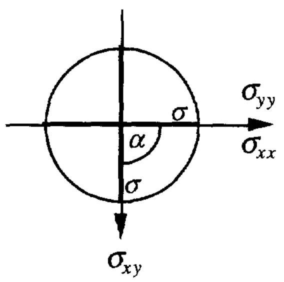


Fig. 6. Mohr's circle corresponding to Fig. 5 with $\alpha = 2\times 45°$.

## 7. PLANE RIGID IDEAL PLASTICITY

We consider a homogeneous deformation, in which the principal directions of the generalized finite strains with respect to the reference configuration $\kappa_0$ change, in contrast to the previous examples in which $\dot{\underset{\sim}{\varepsilon}} = \underset{\sim}{D}$ and eqn (13) hold. We restrict ourselves to a constitutive model of rigid ideal plasticity [see Casey (1986)] where the logarithmic total and plastic strain are identical

$$\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}^p \tag{28}$$

and the hardening slope vanishes, $h = 0$, cf. Fig. 1. As a consequence of eqn (28) and the plastic incompressibility we have

$$\varepsilon_{kk} = D_{kk} = 0. \tag{29}$$

The yield function is rewritten from eqn (16) multiplied by a factor of $2G$

$$f(\underset{\sim}{\sigma}', \bar{\sigma}) = \|\underset{\sim}{\sigma}'\| - \sqrt{\tfrac{2}{3}}\bar{\sigma}. \tag{30}$$

The spherical part of the stress, which is three times the negative hydrostatic pressure, becomes indeterminate. The stress deviator can be determined as long as flow occurs: its magnitude follows in this case from the yield function eqn [(30)] and the yield condition†

$$f(\underset{\sim}{\sigma}', \bar{\sigma}) = 0, \tag{31}$$

and its direction follows from the flow rule. But the question of interest is: which‡ flow rule? Is it the Cauchy stress deviator proportional (denoted by $\sim$) to the deviatoric rate of deformation tensor, or is it the work-conjugate stress deviator proportional to the deviatoric logarithmic strain rate

$$\hat{\underset{\sim}{T}}' \sim \hat{\underset{\sim}{D}} \quad \text{or} \quad \hat{\underset{\sim}{\sigma}}' \sim \overset{\Delta}{\hat{\varepsilon}}? \tag{32}$$

It is obvious, that by a simple rotation of the above deviators, eqns (32) are equivalent to

$$\underset{\sim}{T}' \sim \underset{\sim}{D} \quad \text{or} \quad \underset{\sim}{\sigma}' \sim \dot{\underset{\sim}{\varepsilon}}. \tag{33}$$

Prandtl (1920) and Geiringer (1930) have chosen the first part of eqn (32) as the flow rule. But for the following example we choose the second part of eqn (33) as the flow rule in order to be consistent with our constitutive model.

Consider for example the homogeneous equi-volumetrical plane finite deformation

$$x = (X\cos\varphi - Y\sin\varphi)\mu, \quad y = (X\sin\varphi + Y\cos\varphi)/\mu, \tag{34}$$

so that the components of the deformation gradient $\underset{\sim}{F}$ and the rotation tensor $\underset{\sim}{R}$ are

$$F_{iJ} = \begin{bmatrix} \mu\cos\varphi & -\mu\sin\varphi \\ \dfrac{1}{\mu}\sin\varphi & \dfrac{1}{\mu}\cos\varphi \end{bmatrix}, \quad R_{iI} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix},$$

of the right and left stretch tensors $\underset{\sim}{U}$ and $\underset{\sim}{V}$ are

$$U_{IJ} = \frac{1}{\mu}\begin{bmatrix} \sin^2\varphi + \mu^2\cos^2\varphi & \dfrac{1-\mu^2}{2}\sin(2\varphi) \\ \dfrac{1-\mu^2}{2}\sin(2\varphi) & \mu^2\sin^2\varphi + \cos^2\varphi \end{bmatrix}, \quad V_{ij} = \begin{bmatrix} \mu & 0 \\ 0 & \dfrac{1}{\mu} \end{bmatrix}$$

and of the velocity gradient $\underset{\sim}{G}$ with respect to the current configuration $\hat{\kappa}$ are

$$G_{ij} = \begin{bmatrix} \dfrac{\dot{\mu}}{\mu} & -\mu^2\dot{\varphi} \\ \dfrac{\dot{\varphi}}{\mu^2} & -\dfrac{\dot{\mu}}{\mu} \end{bmatrix}.$$

With the abbreviations

† For isotropic incompressible material all yield conditions for states of plane deformation reduce to a form similar to (30), (31)—see Sayir (1970).

‡ If the reference and current configurations coincide initially, then eqn (13) holds and the difference between the flow rules vanishes—even for infinitesimal deformations.

$$a_{IJ} = \begin{bmatrix} \cos(2\varphi) & -\sin(2\varphi) \\ -\sin(2\varphi) & -\cos(2\varphi) \end{bmatrix}, \quad b_{IJ} = \begin{bmatrix} \sin(2\varphi) & \cos(2\varphi) \\ \cos(2\varphi) & -\sin(2\varphi) \end{bmatrix}, \tag{35}$$

the components of $\underset{\sim}{D}$, $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{\dot{\varepsilon}}$ are

$$D_{IJ} = \frac{\dot{\mu}}{\mu} a_{IJ} + \frac{1-\mu^4}{2\mu^2} \dot{\varphi}\, b_{IJ}, \quad \varepsilon_{IJ} = \ln(\mu) a_{IJ}, \quad \dot{\varepsilon}_{IJ} = \frac{\dot{\mu}}{\mu} a_{IJ} - 2\,\dot{\varphi} \ln(\mu) b_{IJ}. \tag{36}$$

and of $\hat{\underset{\sim}{D}}$, $\hat{\underset{\sim}{\varepsilon}}$ and $\overset{\Delta}{\hat{\underset{\sim}{\varepsilon}}}$ are

$$\hat{D}_{ij} = \begin{bmatrix} \dfrac{\dot{\mu}}{\mu} & \dfrac{1-\mu^4}{2\mu^2}\dot{\varphi} \\ \dfrac{1-\mu^4}{2\mu^2}\dot{\varphi} & -\dfrac{\dot{\mu}}{\mu} \end{bmatrix}, \quad \hat{\varepsilon}_{ij} = \ln(\mu) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \overset{\Delta}{\hat{\varepsilon}}_{ij} = \begin{bmatrix} \dfrac{\dot{\mu}}{\mu} & -2\dot{\varphi} ln(\mu) \\ -2\dot{\varphi} ln(\mu) & -\dfrac{\dot{\mu}}{\mu} \end{bmatrix}, \tag{37}$$

respectively. For the deformation [eqns (34)] the principal direction of the generalized finite strains with respect to $\kappa_0$ is given by the negative rotation angle $\varphi$. The non-zero components of the fourth order transformation tensors with respect to $\kappa_0$ are given by

$$\alpha_{xxxx} = \alpha_{yyyy} = 1 + \frac{1}{2}\frac{\chi}{1-\chi}\sin^2(2\varphi), \quad \beta_{xxxx} = \beta_{yyyy} = 1 - \frac{\chi}{2}\sin^2(2\varphi),$$

$$\alpha_{xxxy} = \alpha_{xyxx} = \frac{1}{2}\frac{\chi}{1-\chi}\sin(2\varphi)\cos(2\varphi), \quad \beta_{xxxy} = \beta_{xyxx} = -\frac{\chi}{2}\sin(2\varphi)\cos(2\varphi),$$

$$\alpha_{xyyy} = \alpha_{yyxy} = -\alpha_{xxxy}, \quad \beta_{xyyy} = \beta_{yyxy} = -\beta_{xxxy},$$

$$\alpha_{xxyy} = \alpha_{yyxx} = -\frac{1}{2}\frac{\chi}{1-\chi}\sin^2(2\varphi), \quad \beta_{xxyy} = \beta_{yyxx} = \frac{\chi}{2}\sin^2(2\varphi),$$

$$\alpha_{xyxy} = \frac{1}{2}\left[1 + \frac{\chi}{1-\chi}\cos^2(2\varphi)\right], \quad \beta_{xyxy} = \frac{1}{2}[1 - \chi\cos^2(2\varphi)] \tag{38}$$

and with respect to $\hat{\kappa}$ by†

$$\hat{\alpha}_{xxxx} = \hat{\alpha}_{yyyy} = 1, \quad \hat{\beta}_{xxxx} = \hat{\beta}_{yyyy} = 1,$$

$$\hat{\alpha}_{xyxy} = \frac{1}{2(1-\chi)}, \quad \hat{\beta}_{xyxy} = \tfrac{1}{2}[1-\chi]$$

with

$$\chi = 1 + \frac{4\mu^2 \ln(\mu)}{1-\mu^4}.$$

For $\mu = 1$ the latter factor $\chi$ vanishes so that eqn (13) holds.

We specify the two kinematic parameters

† Only non-zero components are specified, and of course, the symmetries [eqn (9)] hold.

$$\mu = \exp(\tau), \quad \varphi = \tau$$

by functions of the monotonically increasing or decreasing load parameter $\tau$, such that eqns (36) and (37) become

$$D_{IJ} = \dot{\tau} a_{IJ} - \dot{\tau} \sinh(2\tau) b_{IJ}, \quad \dot{\varepsilon}_{IJ} = \dot{\tau} a_{IJ} - 2\tau \dot{\tau} b_{IJ}, \tag{39}$$

and

$$\hat{\mathrm{D}}_{ij} = \dot{\tau} \begin{bmatrix} 1 & -\sinh(2\tau) \\ -\sinh(2\tau) & -1 \end{bmatrix}, \quad \overset{\Delta}{\varepsilon}_{ij} = \dot{\tau} \begin{bmatrix} 1 & -2\tau \\ -2\tau & -1 \end{bmatrix}, \tag{40}$$

where sinh denotes the *hyperbolic sine function* and the $a_{IJ}$, $b_{IJ}$ are given by eqn (35). The work-conjugate stress deviators with respect to $\kappa_0$ and its components

$$\underset{\sim}{\sigma}' = \sqrt{\frac{2}{3}} \frac{\bar{\sigma}}{\|\dot{\underset{\sim}{\varepsilon}}\|} \dot{\underset{\sim}{\varepsilon}}, \quad \sigma'_{IJ} = (a_{IJ} - 2\tau b_{IJ}) \frac{\bar{\sigma}}{\sqrt{3(1+4\tau^2)}} \tag{41}$$

or with respect to $\hat{\kappa}$

$$\hat{\underset{\sim}{\sigma}}' = \sqrt{\frac{2}{3}} \frac{\bar{\sigma}}{\|\overset{\Delta}{\underset{\sim}{\varepsilon}}\|} \overset{\Delta}{\underset{\sim}{\varepsilon}}, \quad \hat{\sigma}'_{ij} = \begin{bmatrix} 1 & -2\tau \\ -2\tau & -1 \end{bmatrix} \frac{\bar{\sigma}}{\sqrt{3(1+4\tau^2)}}$$

follow from eqns (39) and (40), the flow rule eqn (33) (second part) and the yield condition eqns (30) and (31). The back-rotated stress deviator with respect to $\kappa_0$ and its components

$$\underset{\sim}{\mathrm{T}}' = \underset{\smile}{\beta} \underset{\sim}{\sigma}', \quad T'_{IJ} = \left( a_{IJ} - \frac{4\tau^2}{\sinh(2\tau)} b_{IJ} \right) \frac{\bar{\sigma}}{\sqrt{3(1+4\tau^2)}}$$

or the Cauchy stress deviator with respect to $\hat{\kappa}$ and its components

$$\hat{\underset{\sim}{\mathrm{T}}}' = \hat{\underset{\smile}{\beta}} \hat{\underset{\sim}{\sigma}}', \quad \hat{\mathrm{T}}'_{ij} = \begin{bmatrix} 1 & -\dfrac{4\tau^2}{\sinh(2\tau)} \\ -\dfrac{4\tau^2}{\sinh(2\tau)} & -1 \end{bmatrix} \frac{\bar{\sigma}}{\sqrt{3(1+4\tau^2)}} \tag{42}$$

follow from eqns (15), (38) and (41).

For finite deformations the result [eqn (42)] is different from a result based on the flow rule (32) (first part), which would lead to the Cauchy stress deviator

$$\hat{\underset{\sim}{\mathrm{T}}}' = \sqrt{\frac{2}{3}} \frac{\bar{\sigma}}{\|\hat{\underset{\sim}{\mathrm{D}}}\|} \hat{\underset{\sim}{\mathrm{D}}}, \quad \hat{\mathrm{T}}'_{ij} = \begin{bmatrix} 1 & -\sinh(2\tau) \\ -\sinh(2\tau) & -1 \end{bmatrix} \frac{\bar{\sigma}}{\sqrt{3(1+\sinh^2(2\tau))}},$$

i.e. the example may be reversed by starting with the flow rule eqn (32) (first part) and by using the transformations given in eqns (8), (12), (14) and (15).

## 8. CONCLUSIONS

The logarithmic strain space is well-suited to describing elasto-plastic deformation processes. The trace and deviator of the logarithmic strain tensor keep their physical meaning of dilatation and distortion measures even for the finite deformation theory.

Therefore, constitutive models based on traces and deviators, as they are widely used in infinitesimal deformation theories, can be generalized to finite deformation theories within the framework of the logarithmic strain space description. This is illustrated with the three analytical examples presented.

In the first and second example of finite tension and finite shear the logarithmic strain rates are identical to the rate of deformation tensors, and thus their work-conjugate stresses, since the principal directions of the generalized finite strains with respect to the reference configuration $\kappa_0$ remain unchanged. These two examples demonstrate the additive split of finite dilatation and finite distortion of the logarithmic strain tensor. In the example of finite shear an initially square shape with side lengths $a$ is deformed into a parallelogram with equal side lengths $b$ as depicted in Figs 3 and 5 and *not* into a parallelogram with side lengths having pairs of $a$ and $c$ as depicted in Fig. 4. Since a homogeneous shear stress distribution has two planes of symmetry the deformed shape for an isotropic material also needs to be doubly symmetric as is the case for the parallelograms of Figs 3 and 5. The parallelogram of Fig. 4 definitely does not have two planes of symmetry. In the third example of rigid ideal plasticity the logarithmic strain rates and the rate of deformation tensors differ due to the change of the principal directions of the generalized finite strains with respect to the reference configuration $\kappa_0$. Hence the stress deviators, which are work-conjugate to the deformation rates, are also different. Because the relations between the deformation rates and the corresponding stress deviators are given by the transformations [eqns (8), (12), (14) and (15)] they serve as a link between the classical rigid plasticity flow theory of Prandtl (1920), Geiringer (1930) and the logarithmic strain space description of plasticity.

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